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Dominant matrices and max algebra ☆

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ABSTRACT

We study the class of so-called totally dominant matrices in the usual algebra and in the max algebra in which the sum is the maximum and the multiplication is usual. It turns out that this class coincides with the well known class of positive matrices having positive the determinants of all 2×2 submatrices. The closure of this class is closed not only with respect to the usual but also with respect to the max multiplication. Further properties analogous to those of totally positive matrices are proved and some connections to Monge matrices are mentioned.

1. Introduction

In [4], the so-called *k-subtotally positive matrices* were defined as real matrices all of whose square submatrices of order at most k have positive determinant.

Clearly, if the matrix is $m \times n$, then for $k = 1$, such matrix is positive, if $k = \min(m, n)$, the matrix is totally positive.

As in [6], the *relevant submatrix* of an $m \times n$ matrix A is a square submatrix whose rows as well as columns are consecutive and either the first row, or the first column (or, both) are in the first row or in the first column of A.

Remark 1.1. Observe that to every entry of a square matrix A one can assign exactly one relevant submatrix, namely such that its lower right corner entry is that entry.

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Analogously to the theory of totally positive matrices, the following result was proved in [4] which shows that for an $m \times n$ matrix, only mn inequalities suffice for distinguishing whether a real matrix is a k-subtotally positive matrix, if k is fixed, and, what may be interesting, independently of k, $1 \leqslant k \leqslant \min(m, n)$.

Theorem A ([4, Theorem 2.1])**.** *Let A be a real $m \times n$ matrix, let k be an integer, $1 \leqslant k \leqslant \min(m, n)$. Then, A is k-subtotally positive if and only if for all j, $1 \leqslant j < k$, all $j \times j$ relevant submatrices have positive determinant, and, in addition, all $k \times k$ submatrices of A with consecutive rows and consecutive columns have positive determinant.*

A simple application of the Cauchy–Binet theorem implies the following.

Theorem B ([4], Theorem 2.2)**.** *The product of k-subtotally positive matrices (which can be multiplied) is also a k-subtotally positive matrix.*

Let us also recall the following factorization theorem from [6]:

Theorem C. *An $n \times n$ real matrix A is totally positive if and only if it can be expressed as*

$$A = B_1 B_2 \cdots B_{n-1} D C_{n-1} C_{n-2} \cdots C_1, \tag{1}$$

where for $i = 1, \ldots, n-1$, B_i is the lower bidiagonal matrix

$$B_i = \begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & 0 & 1 & & & \\ & & & \alpha_{n-i+1,1} & 1 & & \\ & & & & \ddots & \ddots & \\ & & & & & \alpha_{ni} & 1 \end{bmatrix}, \tag{2}$$

and C_i is the upper bidiagonal matrix

$$C_i = \begin{bmatrix} 1 & 0 & & & & & \\ & \ddots & \ddots & & & & \\ & & 1 & 0 & & & \\ & & & 1 & \beta_{1,n-i+1} & & \\ & & & & \ddots & \ddots & \\ & & & & & 1 & \beta_{in} \\ & & & & & & 1 \end{bmatrix}, \tag{3}$$

D is a diagonal matrix, and all the numbers α_{ik}, $i > k$, β_{ik}, $i < k$, and all the diagonal entries of D are positive.

Remark 1.2. The numbers α_{ik}, β_{ik} and the diagonal entries are by the matrix A uniquely determined.

2. Results

In this section, we call a real square matrix *dominant* if it is nonnegative and if the product of its diagonal entries is greater than such product of any matrix obtained by multiplication of the matrix from the right by any nontrivial permutation matrix. A 1×1 matrix $[\,c\,]$ will be called dominant if c is positive.

We call a real matrix (not necessarily square) *totally dominant* if all its square submatrices are dominant.

Theorem 2.1. *Every totally dominant matrix is 2-subtotally positive.*

Proof. Let A be totally dominant. Then every 2×2 submatrix of A is totally positive so that A is 2-subtotally positive. $\square$

Remark 2.2. The example $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 1 & 5 & 15 \end{bmatrix}$ shows that for $n > 2$ not every totally dominant matrix, even symmetric, need not be totally positive.

We now prove a lemma.

Lemma 2.3. *Let A be a square 2-subtotally positive matrix. Then there exist diagonal matrices D_1 and D_2 with positive diagonal entries, such that D_1AD_2 has all diagonal entries equal to one and all off-diagonal entries smaller than one.*

Proof. We can assume that A has at least 3 rows. Denote by $\mathcal{T}_A$ the set of all matrices of the form D_1AD_2, when D_1 and D_2 are diagonal matrices with positive diagonal entries, and such that D_1AD_2 has all diagonal entries one. We shall show that the minimum m of the maximal off-diagonal entries of all matrices in $\mathcal{T}_A$ is smaller than one. Such minimum clearly exists and is attained for some matrix $B = [b_{ik}]$. Let, say $b_{pq} = m$, $p < q$. This entry cannot be the only entry of B equal to m since the multiplication of the pth row by a number τ slightly smaller than one and multiplication of the pth column by $1/\tau$ diminishes this minimum m. Suppose that the entry b_{pq} is the entry with the smallest p which is equal to m. As before, in the pth column there is an entry b_{rp}, also equal to m. By the minimality property of p, $r > p$. The determinant of the 2×2 submatrix with rows p and r and columns p and q is positive so that $b_{rq} > m^2$ which indeed implies $m < 1$. $\square$

Before we prove the next basic theorem, we shall call the product corresponding to a term of the determinant of a square matrix A a *transversal product* of A. It is clear that every transversal product of a submatrix of A can be completed to a transversal product of A by some transversal product in the complement of the submatrix in A.

Theorem 2.4. *Let A be an $m \times n$ real matrix. Then the following are equivalent:*

(1) A is a totally dominant matrix.
(2) A is a 2-subtotally positive matrix.

Proof. By Theorem 2.1, 1.implies 2. To prove the converse, let A satisfy 2. It suffices to prove that every square submatrix of A is dominant. Call it A_1. By Lemma 2.3, there exist diagonal matrices D_1 and D_2 with positive diagonal entries such that the matrix $B = D_1A_1D_2$ has all diagonal entries one and all off-diagonal entries smaller than one. This, of course, means that the product of the diagonal entries of B is one and the product of the entries in every other transversal is less than one so that B is dominant. Therefore, A_1 is dominant as well. $\square$

By Theorems B and A,we have:

Corollary 2.5. *The product of totally dominant matrices of compatible sizes is also a totally dominant matrix.*

Corollary 2.6. *Every relevant submatrix of a totally dominant matrix is dominant. If all relevant 2×2 submatrices of a real matrix as well as all the remaining 2×2 submatrices with consecutive rows and consecutive columns are dominant then the matrix is totally dominant.*

It is well known that nonnegative bidiagonal matrices form the multiplicative building stones of the set of totally nonnegative matrices. Apparently, an analogous set of building stones for the class of 2-subtotally positive (or, 2-subtotally nonnegative) matrices is not known.

In the sequel, we shall use the max algebra [2] in which for nonnegative numbers the max sum is defined as $a \oplus b = \max(a, b)$ and the max product is $a \otimes b = a.b$. Observe that the product $A \otimes B$ of two matrices $A = [a_{ij}]$ and $B = [b_{kl}]$ is then $C = [c_{ik}]$, where $c_{ik} = \max_j(a_{ij}b_{jk})$ for all admissible i and k. One can also introduce the notion of the *max permanent*, mp for short, as the maximum of all the products of the entries in the transversals. For a totally dominant square matrix, it is the product of the diagonal entries. The following is easy to prove.

Theorem 2.7. *If A and B are square nonnegative matrices, then for the max permanents,* $\mathrm{mp}(A \otimes B) \geqslant \mathrm{mp}(A)\mathrm{mp}(B)$.

Proof. Let $A = [a_{ij}]$, $B = [b_{kl}]$, $i, j, k, l = 1, \ldots, n$. We have $\mathrm{mp}(A) = a_{1i_1}a_{2i_2} \cdots a_{ni_n}$ for some permutation $(i_1 i_2 \cdots i_n)$ of $1, 2, \ldots, n$, and similarly for B. Denoting $A \otimes B$ as $C = [c_{ik}]$, we obtain for $\mathrm{mp}(C)$ the maximum of terms among which is the product of the terms maximizing those in $\mathrm{mp}(A)$ and $\mathrm{mp}(B)$. $\square$

The whole theory of totally dominant matrices can be extended to the theory of, say, *weakly totally dominant matrices*, in which we admit non-strict inequalities instead of strict inequalities. It is easily seen that they are in the closure of the set of totally dominant matrices of the corresponding size.

In fact, if we define *k-subtotally nonnegative matrices* as such that their every square submatrix with at most k rows has nonnegative determinant, then the following corollary to Theorem 2.4 holds.

Corollary 2.8. *Let A be a real matrix. Then the following are equivalent:*

(1) A is a weakly totally dominant matrix.
(2) A is a 2-subtotally nonnegative matrix.

We also have:

Corollary 2.9. *Every totally positive matrix is totally dominant.*

We are now able to prove the basic result.

Theorem 2.10. *The max product of totally dominant matrices is a weakly totally dominant matrix if the matrices have compatible sizes.*

Proof. Suppose that A and B are totally dominant matrices such that AB exists. We shall show that all square 2×2 submatrices of the max product $C = A \otimes B$ have nonnegative determinant. Of course, the entries c_{ik} of C are all positive so that by Corollary 2.8 it will follow that C is weakly totally dominant.

Let thus $p < q$ be two row indices of C, $r < s$ two column indices. We shall show that $c_{pr}c_{qs} \geqslant c_{ps}c_{qr}$. Suppose that $c_{pr}c_{qs} < c_{ps}c_{qr}$. Since $c_{ps} = \max_i(a_{pi}b_{is})$, $c_{qr} = \max_j(a_{qj}b_{jr})$, let these maxima be attained for $i = i_0, j = j_0$. It follows that whenever k and l are admissible indices, then

$$a_{pk}b_{kr}a_{ql}b_{ls} < a_{pi_0}b_{i_0s}a_{qj_0}b_{j_0r}.$$

Choose first $k = i_0$, $l = j_0$. It follows that $b_{i_0r}b_{j_0s} < b_{j_0r}b_{i_0s}$. Since $r < s$ and B is totally dominant, necessarily $j_0 < i_0$. Then choose $k = j_0$, $l = i_0$. We obtain $a_{pj_0}a_{qi_0} < a_{pi_0}a_{qj_0}$. Since $p < q$ and A is totally dominant, $i_0 < j_0$, a contradiction. $\square$

Remark 2.11. Theorem 2.10 is not correct without the word weakly as can be shown already for a 2×2 example.

By continuity, it follows.

Corollary 2.12. *The max product of weakly totally dominant matrices is a weakly totally dominant matrix if the matrices have compatible sizes.*

Now we shall show that an analogous factorization as for totally positive matrices (1) holds for totally dominant matrices if we use the max algebraic approach. In some detailed proofs, the following two simple lemmata are useful.

Lemma 2.13. *If A and B are nonnegative matrices which can be multiplied, then the usual product AB and the max product $A \otimes B$ have the same zero–nonzero structure.*

Lemma 2.14. *Write a square totally dominant matrix A in the usual way as $A = B + D + C$, where B is the subdiagonal lower triangular part of A (completed by zeros), C the superdiagonal upper triangular part and D the diagonal part of A. Then A is the max algebraic product*

$$A = (I + BD^{-1}) \otimes D \otimes (I + D^{-1}C). \tag{4}$$

Proof. We can assume that the diagonal part D is the identity matrix. Then checking the right-hand side of (4) yields c_{ij} for $i > j$, since $c_{ij} > c_{ik}b_{kj}$ whenever $k < i$, b_{ij} for $i < j$, since $b_{ij} > c_{ik}b_{kj}$ whenever $k < j$; for $i = j$, $a_{ii} = 1$ since $c_{ik}b_{ki} < 1$ whenever $k < i$. $\square$

Remark 2.15. The Eq. (4) shows that the *LU* decomposition of a totally dominant matrix is in max algebra very simple, in particular for $D = I$. The matrix $I + BD^{-1}$ can be called, for a moment, a *unit lower triangularly totally dominant matrix* since all submatrices in the lower triangular part are dominant.

Lemma 2.16. *Every unit lower triangularly totally dominant matrix $B = [b_{ik}]$ is a max product $B_1 \otimes B_2 \otimes \cdots \otimes B_{n-1}$, where the matrices B_i are as in (2) and the αs satisfy $\alpha_{ki} = \frac{b_{ki}}{b_{k-1,i}}$, $1 \leqslant i < k \leqslant n$.*

Proof. Follows by an easy induction with respect to n and observing that always $\max(\alpha_{k,l-1}, \alpha_{kl}) = \alpha_{kl}$, whenever $1 < l < k \leqslant n$. $\square$

Theorem 2.17. *Let $A = [a_{ik}]$ be an $n \times n$ matrix obtained as max product of the matrices in (2) and (3). Then A is a positive weakly totally dominant matrix. On the other side, every square totally dominant matrix can be written as such max product of matrices as in (1)–(3). One such factorization is obtained by choosing the numbers α_{ik}, β_{ik}, and the diagonal entries d_i of D as*

$$d_i = a_{ii}, \quad \text{for} \quad i < k, \quad \alpha_{ki} = \frac{a_{ki}}{a_{k-1,i}}, \quad \beta_{ik} = \frac{a_{ik}}{a_{i,k-1}}. \tag{5}$$

Proof. Since all the factors in the product are weakly totally dominant matrices, the max product A is weakly totally dominant by Corollary 2.10. It thus suffices to show that A is positive. This follows from Lemma 2.13 since both the max products $B_1 \cdots B_{n-1}$ and $C_{n-1} \cdots C_1$ have positive lower and upper triangular parts.

The converse follows from Lemma 2.14 and from Lemma 2.16 applied for the matrix $I + BD^{-1}$ and for the transpose of $I + D^{-1}C$ with appropriate changes. $\square$

3. Concluding remarks

We could generalize the results for non-square and even weakly totally dominant matrices by including nonnegative bidiagonal matrices into the max product. The technique of proofs for positive matrices is easy by observing that a positive weakly totally dominant matrix will become totally positive using the Hadamard (entrywise) multiplication by an appropriate totally dominant matrix close to the matrix of all ones, say, $\left[1 + \frac{\varepsilon}{i+j-1}\right]$, and letting the positive small ε converge to zero.

Also applications to the (0, 1)-matrices (which are in max algebra closed with respect to max multiplication and max addition) seem to be interesting. Let us mention that in the case of *intrinsic factorization* of matrices introduced in [5] the max product coincides (say, for nonnegative matrices) with the usual product. This happens among other cases for the *complementary basic matrices* introduced in [3].

There is also a close relationship between the 2-subtotally positive matrices and the (strict anti-) Monge matrices [4] given by the logarithms of entries of the 2-subtotally (thus totally dominant) positive matrices. Observe thus that by Corollary 2.8, it follows that the anti-Monge matrix, i.e., a real matrix $[c_{ik}]$ for which $c_{ik} + c_{jl} \geqslant c_{il} + c_{jk}$ whenever $i < j$ and $k < l$, has the maximal trace among all sums of entries in the transversals. The max algebraic properties of the totally dominant matrices correspond then to properties of the Monge (or, anti-Monge) matrices in the "usual" max algebra for which $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$ for all real a and b (cf. [1]).

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